Operations on Power Series Related to Taylor Series

In this problem, we perform elementary operations on Taylor series – term by term differentiation and integration – to obtain new examples of power series for which we know their sum. Suppose that a function f has a power series representation of the form:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

convergent on the interval $(c-R, c+R)$ for some R. The results we use in this example are:

- (Differentiation) Given f as above, $f'(x)$ has a power series expansion obtained by by differentiating each term in the expansion of $f(x)$:

$$f'(x) = a_1 + a_2(x-c) + 2a_3(x-c) + \cdots = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

- (Integration) Given f as above, $\int f(x)\,dx$ has a power series expansion obtained by by integrating each term in the expansion of $f(x)$:

$$\int f(x)\,dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  for some constant C depending on the choice of antiderivative of f.

Questions:

1. Find a power series representation for the function $f(x) = \arctan(5x)$. (Note: $\arctan x$ is the inverse function to $\tan x$.)

2. Use power series to approximate
$$\int_0^1 \sin(x^2)\,dx$$
(Note: $\sin(x^2)$ is a function whose antiderivative is not an elementary function.)

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$$\int_0^1 \sin(x^2)\,dx$$

(Note: $\sin(x^2)$ is a function whose antiderivative is not an elementary function.)

1. $\frac{d}{dx}\arctan(5x) = \frac{5}{1+25x^2}$

$$= 5\sum_{n=0}^{\infty}(-25x^2)^n$$

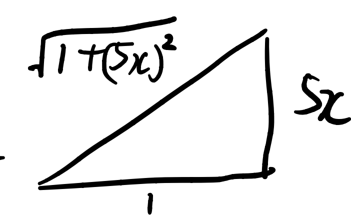
$\arctan(5x) =$

$$= 5(1 - 5^2x^2 + 5^4x^4 + \cdots)$$

$$\arctan(5x) = \int 5(1-5^2x^2+5^4x^4+\cdots)\,dx$$

$$= = 5\left(C_1 + x - \frac{5^2x^3}{3} + \frac{5^4x^5}{5} + \cdots\right)$$

$\tan y = 5x$



$\sec^2 y \frac{dy}{dx} = 5$

$$\frac{dy}{dx} = \frac{5}{\sec^2 y} = \frac{5}{1+25x^2}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{2n+1}$$

when $x=0$, $\arctan 0 = 0 \Rightarrow C = 0$

$$\therefore \arctan(5x) = \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{2n+1}$$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$$

$f(x) = \sin x^2$

$f'(x) = 2x \cos x^2$

$f''(x) = 2\cos x^2 + 2x \cdot 2x(-\sin x^2)$

$$\int_0^1 \sin x^2 \, dx = \int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots dx$$

$$= \left. \frac{x^3}{3} - \frac{x^7}{3! \cdot 7} + \frac{x^{11}}{5! \cdot 11} + \cdots \right|_0^1$$

$$= \left. \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+3}}{(2n+1)! \, 4n+3} \right|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \, 4n+3}$$

$$\Rightarrow \int_0^1 \sin x^2 \, dx \approx \frac{1}{3} - \frac{1}{6 \cdot 7}$$

$$= \frac{14}{42} - \frac{1}{42}$$

$$= \frac{13}{42}$$